Girraween High School

2022

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Total Marks: 100

Section 1 (Pages 2 – 5) **10 Marks**

- Attempt Q1 - Q10
- Allow about 15 minutes for this section

Section 2 (Pages 6-11) **90 marks**

- Attempt Q11 - Q16
- Allow about 2 hours and 45 minutes for this section

General Instructions

- Reading time: 10 minutes
- Working time: 3 Hours
- Write using a black or blue pen
- Board approved calculators may be used
- Laminated reference sheets are provided
- Answer multiple-choice questions by completely colouring in the appropriate circle on your multiple-choice answer sheet.
- Answer questions 11-16 in the appropriate answer booklet and show all relevant mathematical reasoning and/or calculations.

Section 1 (10 marks)

Attempt Questions 1-10

Allow about 15 minutes for this section

Question 1

The converse of "if it miaows it's a cat" is

(A) If it's a cat it miaows. **(B)** If it's not a cat it doesn't miaow.

(C) If it doesn't miaow it isn't a cat. **(D)** If it's a cat it doesn't miaow.

Question 2

The contrapositive of "if it miaows it's a cat" is

(A) If it's a cat it miaows. **(B)** If it's not a cat it doesn't miaow.

(C) If it doesn't miaow it isn't a cat. **(D)** If it's a cat it doesn't miaow.

Question 3

The straight line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ has symmetric (or Cartesian) equation

(A) $x - 1 = y - 4 = \frac{z-3}{2}$ **(B)** $x - 1 = y - 4 = \frac{z+3}{2}$

(C) $1 - x = y - 4 = \frac{z-3}{2}$ **(D)** $1 - x = y - 4 = \frac{z+3}{2}$

Multiple choice continues on the following page

Question 4

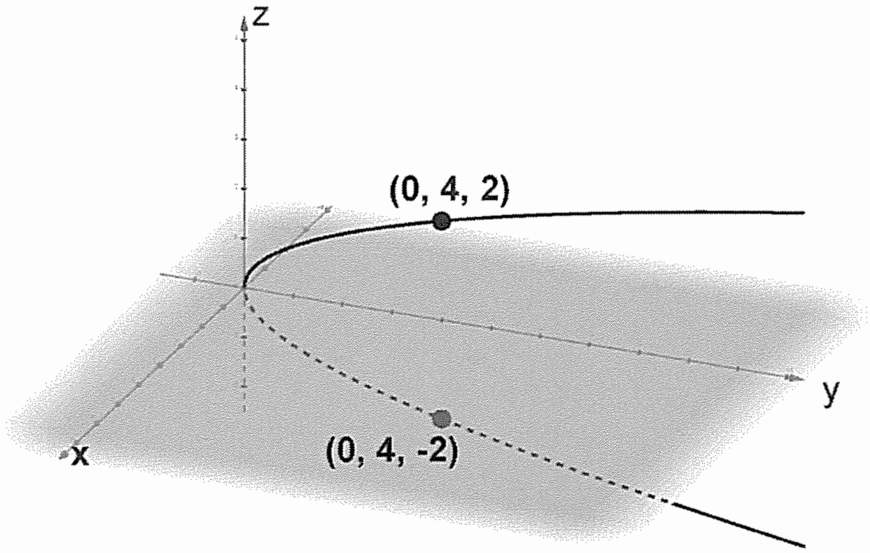
The equation of the curve in 3 dimensional space below is

(A) $x = t^2, y = t, z = 0$

(B) $x = 0, y = t^2, z = t$

(C) $x = 0, y = t, z = t^2$

(D) $x = t^2, y = t^2, z = 0$



Question 5

$$\int \frac{1}{\sqrt{6x - x^2}} . dx =$$

(A) $sin^{-1}\left(\frac{x}{\sqrt{6}}\right) + C$

(B) $sin^{-1}\left(\frac{x}{6}\right) + C$

(C) $sin^{-1}\left(\frac{x-3}{\sqrt{6}}\right) + C$

(D) $sin^{-1}\left(\frac{x-3}{3}\right) + C$

Question 6

If $f(x)$ is even then $\int_{-a}^{a} f(x). dx =$

(A) $2\int_0^a f(a - x). dx$

(B) $2\int_0^{-a} f(a - x). dx$

(C) $2\int_a^0 f(x). dx$

(D) All of the above.

Multiple choice continues on the following page

Multiple choice (continued)

Question 7

$z^2 + pz + q = 0, p, q\ real$ has one solution $z = 3 - 2i$. The values of p and q are:

(A) $p = 6, q = 13$

(B) $p = -6, q = 13$

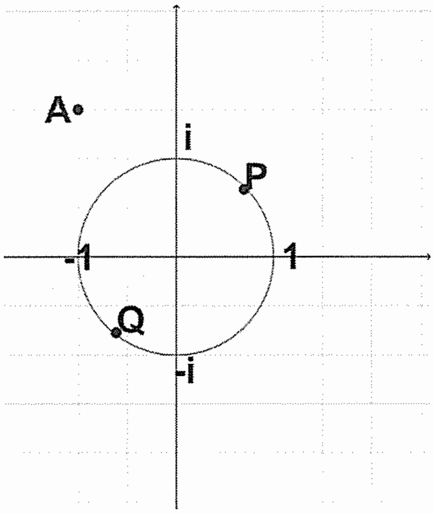
(C) $p = -6, q = -13$

(D) $p = -6, q = -13$

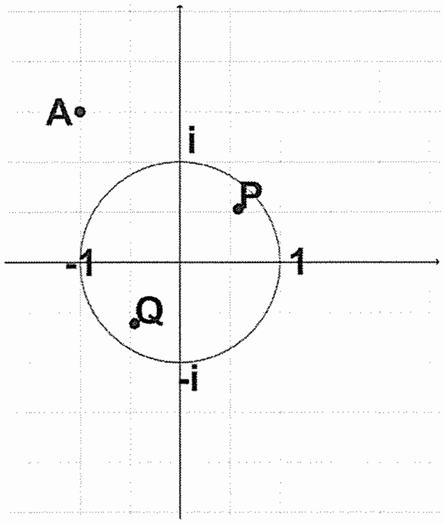
Question 8

If the complex number z has $|z| > 1$ and $\frac{\pi}{2} < Arg(z) < \pi$ and $\overrightarrow{OA} = z$, then if the two square roots of z are $\overrightarrow{OP}\ and\ \overrightarrow{OQ}$, P and Q could be at:

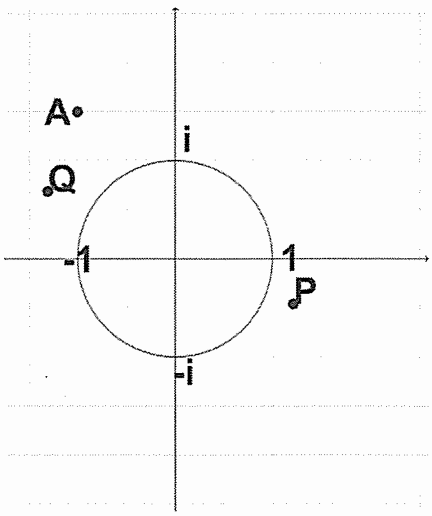
(A)



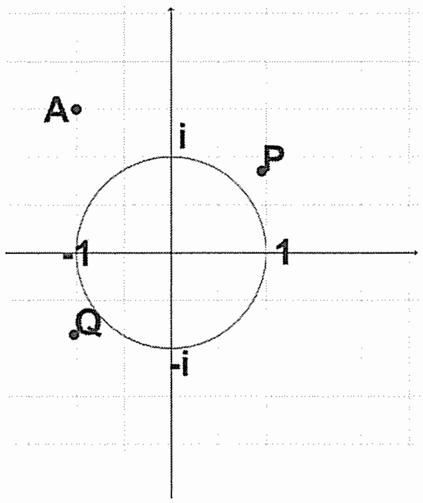
(B)



(C)



(D)



Multiple choice continues on the following page

Question 9

The acceleration of a particle is given by $\ddot{x} = sec^2 x$, where x is the displacement in metres. The rule for the velocity could be:

(A) $v = \sqrt{tan\,x}$

(B) $v = \sqrt{2tan\,x}$

(C) $v = \; tan\,x$

(D) $v = \; 2\,tan\,x$

Question 10

A particle moves with Simple Harmonic Motion (SHM) with equation $x = a\,sin\,nt$. It first RETURNS to its starting point after $\frac{\pi}{3}$ seconds and is travelling at 15m/s forwards at this time. The displacement of the particle of the particle is given by

(A) $x = \frac{5}{2} sin6t$

(B) $x = -\frac{5}{2} sin6t$

(C) $x = 5sin3t$

(D) $x = -5sin3t$

Examination continues on the following page

Section II (90 marks)

Attempt Questions 11-16

Allow about 2 hours and 45 minutes for this section

Start the answers to each question on a separate page in your answer booklet.

In Questions 11-16 your responses should include all relevant mathematical reasoning and/ or calculations.

Question 11 ***(15 marks)*** **Marks**

(a) **(i)** Find $\frac{-1+i\sqrt{3}}{1+i}$ in Cartesian form. **2**

(ii) Express $-1+i\sqrt{3}$ and $1+i$ in modulus/ argument form. **2**

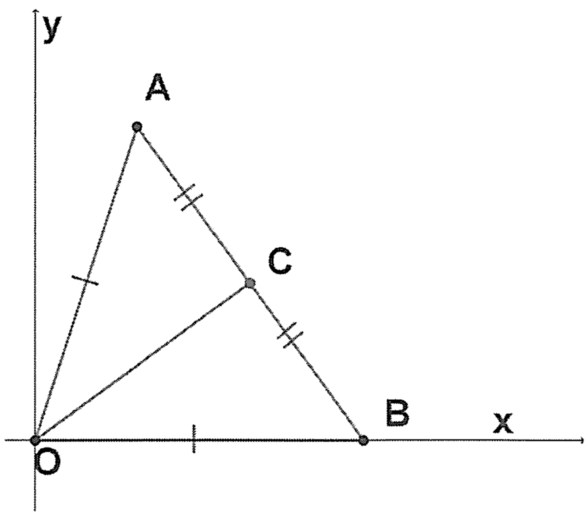
(iii) Hence find the exact value of $tan\ \frac{5\pi}{12}$. **2**

(b) **(i)** By letting $(x+iy)^2 = 7+24i, x, y\ real,$ find $\sqrt{7+24i}$ in Cartesian form. **3**

(ii) Hence solve the equation $z^2 - 3iz - (4+6i) = 0$. **2**

(c) In the Argand diagram below, $\overrightarrow{OA} = z_1, \overrightarrow{OB} = z_2, OA = OB\ and\ \overrightarrow{AC} = \overrightarrow{CB}.$ **2**

Express $\overrightarrow{OC}$ in terms of z_1 and z_2.



(d) Using DeMoivre's Theorem, find the 3 cube roots of $-4\sqrt{2}+4i\sqrt{2}$. **2**

Leave your answers in modulus/argument form.

Examination continues on the following page

Question 12 *(15 marks)* **Marks**

(a) Find $\int sin^{-1}(x).dx$ **3**

(b) Find $\int \frac{1}{2+cos\,x}\,.dx$ **3**

(c) **(i)** Find $A, B\ and\ C$ so that $\frac{A}{x-1}+\frac{Bx+C}{x^2+4}=\frac{-7x-18}{(x-1)(x^2+4)}$. **3**

(ii) Hence find $\int \frac{-7x-18}{(x-1)(x^2+4)}.dx$ **2**

(d) **(i)** Using the substitution $x = sec\,\theta$ or otherwise show by integrating **2**

that $\int \frac{1}{\sqrt{x^2-1}}.dx = ln(x+\sqrt{x^2-1})$

(ii) Find $\int \sqrt{\frac{x-1}{x+1}}.dx$ **2**

Question 13 *(15 marks)*

(a) Prove if a is divisible by 5, a^2-5a is divisible by 50. **4**

(You may assume that the product of two consecutive numbers is even).

(b) **(i)** Prove by contradiction that $\sqrt{2}$ is irrational. **2**

(ii) Prove by contraposition that if n^2 is not divisible **2**

by 18, n is not divisible by 6.

(iii) Using (i), prove that $\sqrt{18}$ is irrational. **1**

(c) **(i)** Prove for all real a and b that $a^2+b^2 \geq 2ab$. **1**

(ii) Hence prove for all $a \geq 0, a+\frac{1}{a} \geq 2$ **1**

(iii) Hence prove for all $a_1, a_2 \geq 0$, **1**

$$(a_1+a_2)(\frac{1}{a_1}+\frac{1}{a_2}) \geq 4$$

(iv) Hence prove by induction for all $a_1, a_2, a_3 ... a_n \geq 0$ **3**

$$(a_1+a_2+a_3+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}\right) \geq n^2$$

(Note: you have already proven this for n = 1 and n = 2)

Examination continues on the following page

Question 14 ***(15 marks)*** **Marks**

(a) A particle is moving so that $v^2 = n^2(a^2 - x^2)$.

(i) Show that the particle is moving in Simple Harmonic Motion. **1**

(ii) If the particle is moving at $\sqrt{15}m/s$ when it is $2m$ to the right of the centre of motion and at $2\sqrt{3}m/s$ when it is 4m to the right of the centre of motion, find the period and amplitude of the motion. **3**

(b) A particle is moving so that $x = 2\,sin3t - 2\sqrt{3}\cos 3t$.

(i) By expressing the motion of the particle in the form $A\,sin(3t - \alpha)$, find the period and amplitude of the motion. **3**

(ii) Find the initial position and velocity of the particle. **2**

(c) A 20kg projectile is launched upwards from the ground at $500m/s$. It experiences force from gravity of 200 Newtons and air resistance in the opposite direction to its motion of $\frac{v^2}{18}$ Newtons.

Letting the ground be $x = 0$,

(i) Show that the weight's acceleration is given by $\ddot{x} = -10 - \frac{v^2}{360}$ **1**

(ii) Show that $x = 180\,ln\left(\frac{253\,600}{3600 + v^2}\right)$ and find the maximum height the projectile reaches. **3**

(iii) Show that the time taken for the projectile to reach a velocity of $v\ m/s$ is given by $t = 6tan^{-1}\left(\frac{25}{3}\right) - 6\,tan^{-1}\frac{v}{60}$ and find the time taken for the projectile to reach its maximum height. **2**

Examination continues on the following page

Question 15 ***(15 marks)*** **Marks**

(a) **(i)** Show *by finding it* that the only point of intersection between **3**

the straight line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and the sphere

$(x-5)^2 + (y+2)^2 + (z-3)^2 = 38$ is the point $\begin{pmatrix} 7 \\ 1 \\ 8 \end{pmatrix}$.

(ii) Show *by finding it* that the point of intersection between the lines **3**

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ -1 \\ 6 \end{pmatrix} + \delta \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$ is also the point

$\begin{pmatrix} 7 \\ 1 \\ 8 \end{pmatrix}$.

(iii) Show that the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ -1 \\ 6 \end{pmatrix} + \delta \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$ is also a tangent to **2**

the sphere $(x-5)^2 + (y+2)^2 + (z-3)^2 = 38$ by showing that it is

perpendicular to the radius of the sphere at $\begin{pmatrix} 7 \\ 1 \\ 8 \end{pmatrix}$.

(iv) Show that the straight line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 11 \end{pmatrix} + \gamma \begin{pmatrix} 6 \\ 1 \\ -3 \end{pmatrix}$ is also a **3**

tangent to the sphere $(x-5)^2 + (y+2)^2 + (z-3)^2 = 38$ by finding

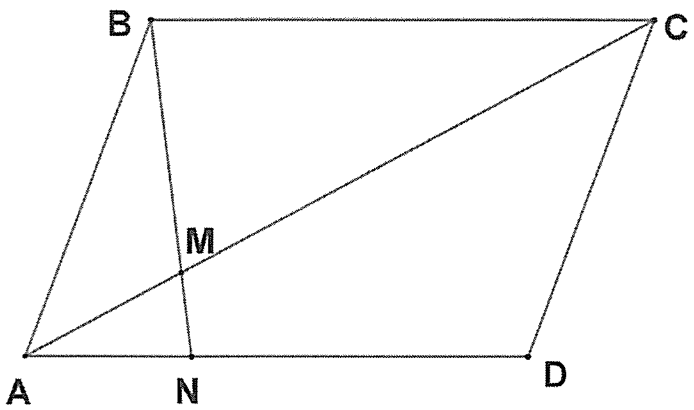
the perpendicular distance from the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 11 \end{pmatrix} + \gamma \begin{pmatrix} 6 \\ 1 \\ -3 \end{pmatrix}$

to the centre of the sphere.

Question 15 continues on the following page

Question 15 (continued) **Marks**

(b) ABCD is a parallelogram. N is on AD so that $\overrightarrow{AN} = \frac{1}{3}\overrightarrow{AD}$. M is on the diagonal AC. *(See diagram).*



(i) Letting $\overrightarrow{AB} = \underline{p}$ and $\overrightarrow{AD} = \underline{q}$, express the vector $\overrightarrow{NB}$ in terms of $\underline{p}$ and $\underline{q}$. **1**

(ii) By letting $\overrightarrow{AM} = \lambda\overrightarrow{AC}$ and $\overrightarrow{MB} = \delta\,\overrightarrow{NB}$, prove that $\overrightarrow{AM} = \frac{1}{4}\overrightarrow{AC}$ **3**

Question 16 *(15 marks)*

(a) **(i)** Using Euler's theorem, show that if $z = e^{i\theta} = \cos\theta + i\sin\theta$ then $z^n + z^{-n} = 2\cos n\theta$. **1**

(ii) Hence show that **2**

$$\cos^7\theta = \frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$$

(iii) Hence find $\int_0^{\frac{\pi}{2}} \cos^7\theta . d\theta$ **2**

(b) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n\theta . d\theta$

(i) Show that $I_n = \frac{n-1}{n} I_{n-2}$ **2**

(ii) Hence find $\int_0^{\frac{\pi}{2}} \cos^7\theta . d\theta$ **2**

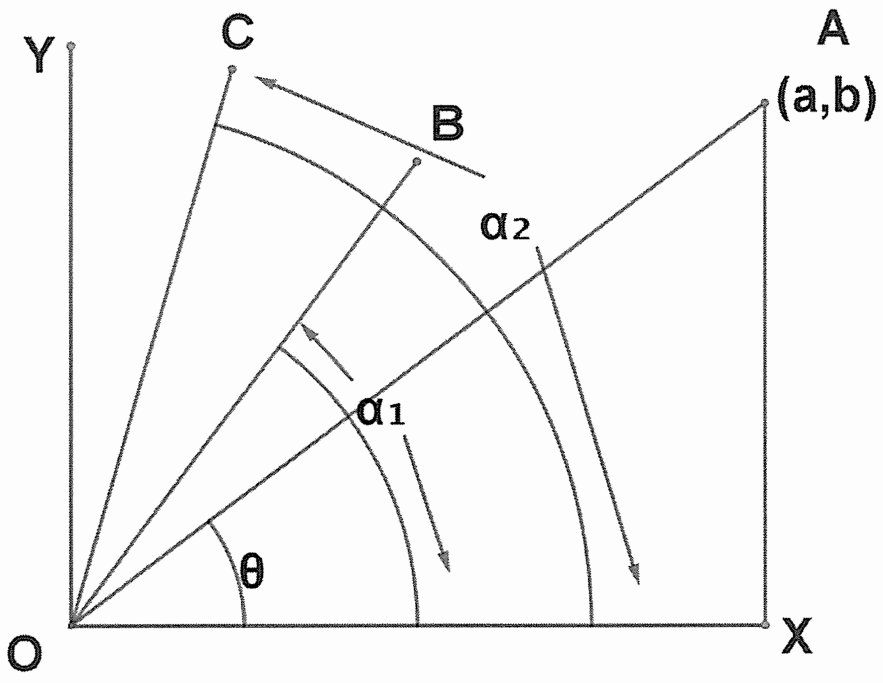
Question 16 continues on the following page

Question 16 (continued) **Marks**

(c) A projectile is launched from O at $V\ m/s$ so that it will hit the point $A\ (a, b)$ on the slope. Air resistance is neglected and the acceleration due to gravity is $g\ m/s^2$.

OX is horizontal so that $\angle AOX = \theta$.

$\angle XOB = \alpha_1$ and $\angle XOC = \alpha_2$ so that α_1 and α_2 are the two angles with the horizontal the projectile can be fired at so that it hits the target at A. *(see diagram).*



(i) Given that $y = \frac{-gx^2}{2V^2}(1 + tan^2\alpha) + x\ tan\ \alpha$, and given (as shown in the diagram) that $\alpha_1 < \alpha_2$, show that **2**

$tan\ \alpha_1 = \frac{V^2 - \sqrt{V^4 - g^2a^2 - 2V^2gb}}{ga}$ and

$tan\ \alpha_2 = \frac{V^2 + \sqrt{V^4 - g^2a^2 - 2V^2gb}}{ga}$

(ii) By showing that $tan\ (\alpha_1 + \alpha_2) = tan\ (90^o + \theta)$, show that $\angle COY = \angle BOA$. **4**

END OF EXAMINATION!!!

GIRRAWEEN HIGH SCHOOL

MATHEMATICS EXTENSION 2
2022 TRIAL HIGHER SCHOOL CERTIFICATE

Student Number: Solutions!

This Booklet contains the answer sheet for Section 1 and Writing Booklet for Section 2.

Section 1 ANSWER SHEET

Select the alternative A, B, C or D that best answers the question.

1.	A	●	B	○	C	○	D	○
2.	A	○	B	●	C	○	D	○
3.	A	○	B	○	C	●	D	○
4.	A	○	B	●	C	○	D	○
5.	A	○	B	○	C	○	D	●
6.	A	●	B	○	C	○	D	○
7.	A	○	B	●	C	○	D	○
8.	A	○	B	○	C	○	D	●
9.	A	○	B	●	C	○	D	○
10.	A	○	B	○	C	○	D	●

Instructions

- If you need more paper for Section 2, please ask your supervisor.
- Write your student number on every booklet you use.
- Write on both sides of each sheet of paper.

Total number of booklets used _______________.

(1) (A) (2) B (3) C (4) B (5) D (6) A (7) B
(8) D (9) B (10) D

(1) A If it's a cat it miaows. $\Rightarrow$ From If A then B to If B then A.
(2) B If it isn't a cat it doesn't miaow. $\Rightarrow$ From if A then B to If not B then not A.

(3) $x = 1 - \lambda \quad y = 4 + \lambda \quad z = 3 + 2\lambda$
$\therefore \lambda = 1 - x. \quad \therefore \lambda = y - 4. \quad \therefore \lambda = \dfrac{z-3}{2}.$

$\therefore 1 - x = y - 4 = \dfrac{z-3}{2}.$ C

(4) B $[x=0,\ y=t^2,\ z=t]$.

(5) $\displaystyle\int \frac{1}{\sqrt{6x - x^2}}\,dx$
$\displaystyle = \int \frac{1}{\sqrt{9-(x-3)^2}}\,dx$
$\displaystyle = \sin^{-1}\left(\frac{x-3}{3}\right) + C.$ D

(6) $\displaystyle\int_{-a}^{a} f(x)\,dx$
$\displaystyle = 2\int_0^a f(x)\,dx$, $f(x)$ even
$\displaystyle = 2\int_0^a f(a-x)\,dx$ A

(7) $\alpha = 3 - 2i$ solution
$\beta = 3 + 2i$ solution
(Conjugate root theorem).
$\alpha + \beta = -p = 6$
$p = -6.$
$\alpha\beta = q = 13$ B
$p = -6, q = 13.$

(8) One root in Q_1.
Other in Q_3
$|\sqrt{z}| > 1$ as $|z| > 1$.
D

(9) $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = \sec^2 x.$
$\dfrac{d}{dx}(v^2) = 2\sec^2 x.$
$v^2 = \displaystyle\int 2\sec^2 x\,dx$
$= 2\tan x + C.$
$v = \sqrt{2\tan x}.$ B

(10) Period $= \dfrac{2\pi}{3} \Rightarrow n = 3.$
$x = a\sin 3t.$
$v = 3a\cos 3t.$
$t = \dfrac{\pi}{3}.\ v = 15.$
$3a\cos\pi = 5.$
$a = -5.$
$x = -5\sin 3t$ D

Q.(11)(a)(i) $\dfrac{-1+i\sqrt{3}}{1+i} \begin{matrix}\times(1-i)\\ \times(1-i)\end{matrix}$

$= \dfrac{(\sqrt{3}-1)}{2} + i\dfrac{(\sqrt{3}+1)}{2}.$

(ii) $-1+i\sqrt{3} = 2\operatorname{cis}\frac{2\pi}{3}.$

$1+i = \sqrt{2}\operatorname{cis}\frac{\pi}{4}.$

(iii) $\dfrac{-1+i\sqrt{3}}{1+i} = \dfrac{2\operatorname{cis}\frac{2\pi}{3}}{\sqrt{2}\operatorname{cis}\frac{\pi}{4}}.$

$\therefore \dfrac{\sqrt{3}-1}{2} + i\dfrac{(\sqrt{3}+1)}{2} = \sqrt{2}\operatorname{cis}\frac{5\pi}{12}.$

Equating parts,

$\dfrac{\sqrt{3}+1}{2} = \sqrt{2}\sin\frac{5\pi}{12}$ (1)

& $\dfrac{\sqrt{3}-1}{2} = \sqrt{2}\cos\frac{5\pi}{12}$ (2)

(1) ÷ (2) $\tan\frac{5\pi}{12} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}.$

$= 2+\sqrt{3}.$

(b)(i) $(x+iy)^2 = 7+24i$

$(x^2-y^2)+2ixy = 7+24i.$

Equating reals

$x^2-y^2 = 7$ (1)

Equating imaginaries

$2xy = 24.$

$y = \dfrac{12}{x}.$ (2)

Sub. (2) in (1):

$x^2 - \dfrac{144}{x^2} = 7$

$x^4-7x^2-144 = 0$

$(x^2-16)(x^2+9) = 0$

As $x \in R$, $x = \pm 4$.

Sub. $x=4$ in (2): $y = \frac{12}{4} = 3.$ | Sub. $x=-4$ in (2): $y = \frac{12}{-4} = -3.$

$\sqrt{7+24i} = \pm(4+3i).$

(ii) $z^2-3iz-(4+6i)=0$

$z = \dfrac{3i \pm \sqrt{(-3i)^2-4\times1\times-(4+6i)}}{2}$

$= \dfrac{3i \pm \sqrt{7+24i}}{2}$

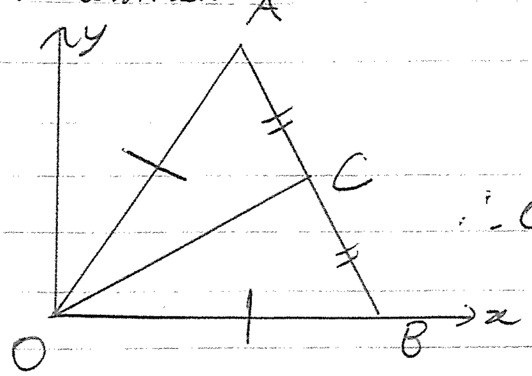
$= \dfrac{3i+(4+3i)}{2}$ or $\dfrac{3i-(4+3i)}{2}$

[using (i)]

$z = 2+3i$ or $z = -2.$

Q. (11) (continued):

(c)



(i) $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

$\therefore \overrightarrow{AC} = \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$.

$\therefore \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB}$

$= \underset{\sim}{a} + \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$

$= \underline{\frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})}$.

(d) $-4\sqrt{2} + 4i\sqrt{2}$

$= 8 \operatorname{cis} \frac{3\pi}{4}$

$\therefore \sqrt[3]{-4\sqrt{2} + 4i\sqrt{2}}$

$= 2 \operatorname{cis} \frac{\pi}{4}$

$= 2 \operatorname{cis} \frac{11\pi}{12}$

$= 2 \operatorname{cis} -\frac{5\pi}{12}$.

Q(12)(a) $\int \sin^{-1}(x).dx$ $\quad u=\sin^{-1}(x) \quad v=x$

$u'=\dfrac{1}{\sqrt{1-x^2}} \quad v'=1$

By $\int u v'.dx = uv - \int v u'.dx$

$$\int \sin^{-1}(x).dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}.dx$$

$$= x\sin^{-1}(x) + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}.dx$$

$$= x\sin^{-1}(x) + \frac{1}{2}\times 2\sqrt{1-x^2} + C \quad \left[\text{By } \int f'(x).[f(x)]^n.dx = \frac{1}{n+1}(f(x))^{n+1}\right]$$

$$= x\sin^{-1}(x) + \sqrt{1-x^2} + C.$$

(b) $\int \dfrac{1}{2+\cos x}.dx$ $\quad t=\tan\left(\frac{x}{2}\right)$ $\quad \dfrac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \dfrac{1+t^2}{2}$ $\quad \therefore dx = \dfrac{2}{1+t^2}.dt$

$$= \int \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}.dt$$

$$= \int \frac{2}{3+t^2}.dt$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right] + C.$$

(c) (i) $\dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+4} = -7x-18$

$\therefore A(x^2+4) + (Bx+C)(x-1) = -7x-18.$

Sub. in $x=1$.

$5A = -25 \Rightarrow A=-5.$

Sub. in $x=0$; $A=-5$.

$-20-C = -18 \Rightarrow C=-2.$

Sub. in $x=2$, $A=5$, $C=-2$.

$-40+(2B-2) = -32.$

$2B-2 = 8$

$B = 5.$

$\underline{A=-5,\ B=5,\ C=-2}$

(ii) Hence

$$\int \frac{-7x-18}{(x-1)(x^2+4)}.dx$$

$$= \int \frac{-5}{x-1} + \frac{5x}{x^2+4} - \frac{2}{x^2+4}.dx$$

$$= -5\ln(x-1) + \frac{5}{2}\ln(x^2+4) - \tan^{-1}\left(\frac{x}{2}\right) + C.$$

Q.(12)(d)(i) $\int \frac{1}{\sqrt{x^2-1}}\,dx$

$x = \sec\theta.$

$dx = \sec\theta\tan\theta.\,d\theta$

$= \int \frac{1}{\tan\theta}.\sec\theta\tan\theta.\,d\theta$

$= \int \sec\theta.\,d\theta$

$= \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}.\,d\theta$

$= \ln[\sec\theta+\tan\theta] + c$

$= \underline{\ln[x+\sqrt{x^2-1}]+c.}$

(ii) $\int \sqrt{\frac{x-1}{x+1}}\,.dx \quad \times\frac{\sqrt{x-1}}{\sqrt{x-1}}$

$= \int \frac{x-1}{\sqrt{x^2-1}}\,.dx$

$= \int \frac{x}{\sqrt{x^2-1}}\,.dx - \int \frac{1}{\sqrt{x^2-1}}\,.dx$

$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}\,.dx - \int \frac{1}{\sqrt{x^2-1}}\,.dx$

$= \underline{\sqrt{x^2-1} - \ln[x+\sqrt{x^2-1}]+c.}$

Q.(13)(a) Either: Let a be divisible by 5 i.e. $a=5k$.

$a^2 - 5a$
$= [5k]^2 - 5\times 5k$
$= 25k^2 - 25k$
$= 25k(k-1)$

As the product of 2 consecutive numbers is even:
Let $k(k-1) = 2m$
so $a^2 - 5a = 25\times 2m$
$= 50m$
which is divisible by 5

OR

Let a be divisible by 5

Case 1 but NOT by 10

$\therefore a = 10k-5, k\in \mathbb{Z}^+$.
a^2-5a
$= (10k-5)^2 - 5(10k-5)$
$= (10k-5)[(10k-5-5]$
$= 5(2k-1)\times 10(k-1)$
$= 50(k-1)(2k-1)$
which is divisible by 50.

Case 2: a divisible by 10:

$a = 10k$.
a^2-5a
$=(10k)^2 - 5(10k)$
$= 100k^2 - 50k$
$= 50k(2k-1)$
which is divisible by 50.

(b)(i) Let $\sqrt{2}$ be rational

i.e. $\sqrt{2} = \frac{p}{q}, p,q \in \mathbb{Z}^+, p,q$ mutually prime.

Squaring BS:
$2 = \frac{p^2}{q^2}$
$2q^2 = p^2 \quad (1)$
$\therefore$ As q NOT factor of p,
2 is a factor of p
$p = 2k$.
$p^2 = 4k^2 \quad (2)$
Sub. (1) in (2):
$2q^2 = 4k^2$
$q^2 = 2k^2$.
$\Rightarrow$ 2 is a factor of q^2
$\therefore$ 2 is a factor of q.
BUT p & q are mutually prime.
Contradiction.
$\sqrt{2}$ is IRRATIONAL.

(ii) Contrapositive.
If n is divisible by 6, n^2 is divisible by 18.
Let $n = 6k$.
$n^2 = 36k^2$
$= 18\times 2k$.
which is divisible by 18.

(iii) $\sqrt{18} = 3\sqrt{2}$.
If $\sqrt{18}$ is rational,
$\sqrt{18} = \frac{p}{q}$
$\therefore \sqrt{2} = \frac{p}{3q}$ which is rational.
But $\sqrt{2}$ is NOT rational.
$\therefore \sqrt{18}$ is irrational.

Q.1(13)(c) (i) If a, b real

$$(a-b)^2 \geq 0$$

$$a^2+b^2 \geq 2ab \quad +2ab.\ BS.$$

(ii) Either using (i)

$$a^2+1 \geq 2a$$

$\div$ BS by a, $a>0$

$$a+\frac{1}{a} \geq 2.$$

or

$$\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2 \geq 0$$

$$a-2+\frac{1}{a} \geq 0$$

$$a+\frac{1}{a} \geq 2.$$

(iii) $(a_1+a_2)\left(\frac{1}{a_1}+\frac{1}{a_2}\right)$

$$= 1+\frac{a_1}{a_2}+\frac{a_2}{a_1}+1$$

$$\geq 1+2+1$$

$$= 4 \quad [\text{as } \frac{a_1}{a_2}+\frac{a_2}{a_1} \geq 2 \text{ by (ii)}].$$

(iv) Assume true for $n=k$

i.e. $(a_1+a_2+\dots+a_k)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_k}\right) \geq k^2$.

Prove true for $n=k+1$

i.e. RTP $(a_1+a_2+\dots+a_k+a_{k+1})\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_k}+\frac{1}{a_{k+1}}\right) \geq (k+1)^2$.

LHS

$$\left[(a_1+a_2+\dots+a_k)+a_{k+1}\right]\left[\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_k}\right)+\frac{1}{a_{k+1}}\right]$$

$$=(a_1+a_2+\dots+a_k)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_k}\right)+\left(\frac{a_1}{a_{k+1}}+\frac{a_2}{a_{k+1}}+\dots+\frac{a_k}{a_{k+1}}\right)+\left(\frac{a_{k+1}}{a_1}+\dots+\frac{a_{k+1}}{a_k}\right)$$

$$+1]$$

$$=(a_1+a_2+\dots+a_k)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_k}\right)+\left[\frac{a_1}{a_{k+1}}+\frac{a_{k+1}}{a_1}+\dots+\frac{a_k}{a_{k+1}}+\frac{a_{k+1}}{a_k}\right]+1$$

$$\geq k^2 + [2+2+\dots+2]+1$$

[By assumption] [using (ii)]

$$= k^2 + 2k + 1$$

$$= (k+1)^2$$

$$= \text{RHS}$$

QED. True by the principle of mathematical induction.

Q. (14) (a) (i) $v^2 = n^2(a^2 - x^2)$

$= n^2a^2 - n^2x^2.$

$\frac{d}{dx}(v^2) = -2n^2x$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2x.$

Particle is moving with SHM.

(ii) $v^2 = n^2(a^2 - x^2)$

$(\sqrt{15})^2 = n^2(a^2 - 2^2)$ | $(2\sqrt{3})^2 = n^2(a^2 - 4^2)$

$15 = n^2a^2 - 4n^2$ (1) | $12 = n^2a^2 - 16n^2$ (2)

(1) − (2): $12n^2 = 3.$

$n^2 = \frac{1}{4},$

$n = \frac{1}{2}.$

$\underline{\text{Period} = 4\pi.}$

Sub. $n = \frac{1}{2}$ in (1):

$15 = \frac{1}{4}(a^2 - 4)$

$\underline{8 = a.}$

$\underline{\text{Amplitude} = 8.}$

(b) (i) $2\sin 3t - 2\sqrt{3}\cos 3t \equiv A\sin(3t - \alpha).$

$2\sin 3t - 2\sqrt{3}\cos 3t \equiv A\sin 3t\cos\alpha - A\cos 3t\sin\alpha$

Equating pats

$2\sin 3t = A\sin 3t\cos\alpha.$ | $2\sqrt{3}\cos 3t = A\cos 3t\sin\alpha.$

$2 = A\cos\alpha$ (1) | $2\sqrt{3} = A\sin\alpha.$

$(1)^2 + (2)^2: A^2[\cos^2\alpha + \sin^2\alpha] = 2^2 + (2\sqrt{3})^2.$

$\underline{A = 4.}$

Sub. $A = 4$ in (1): | Sub. $A = 4$ in (2):

$2 = 4\cos\alpha \Rightarrow \cos\alpha = \frac{1}{2}.$ | $2\sqrt{3} = 4\sin\alpha.$

$\frac{\sqrt{3}}{2} = \sin\alpha.$

$\underline{\alpha = \frac{\pi}{3}.}$

$x = 4\sin\left(3t - \frac{\pi}{3}\right).$ Period $= \frac{2\pi}{3}.$ Amplitude $= 4.$

(ii) Initial position $= 4\sin\left(-\frac{\pi}{3}\right)$ | Initial velocity: $v = 12\cos\left(3t - \frac{\pi}{3}\right)$

$= -2\sqrt{3}$ m | $t = 0$: $v = 12\cos\left(-\frac{\pi}{3}\right) = \underline{6 \text{ m/s}}.$

$2\sqrt{3}$ m LEFT of O.

Q.(14)(a)(i)

$$200N \quad \left(\frac{v^2}{18} N\right)$$

$$F = ma = -200 - \frac{v^2}{18}$$

$$20a = -200 - \frac{v^2}{18}$$

$$\ddot{x} = a = -10 - \frac{v^2}{360}.$$

(ii) $$v.\frac{dv}{dx} = -10 - \frac{v^2}{360}$$

$$= \frac{-3600 - v^2}{360}$$

$$\frac{dv}{dx} = \frac{-3600 - v^2}{360v}$$

$$\frac{dx}{dv} = \frac{-360v}{3600 + v^2}$$

$$x = -180 \int \frac{2v}{3600 + v^2}.dv$$

$$= -180 \ln(3600 + v^2) + C.$$

$x = 0.\ v = 500.$

$$0 = -180 \ln(3600 + 500^2) + C$$

$$180 \ln(253600) = C$$

$$x = 180 \ln\left[\frac{253600}{3600 + v^2}\right].$$

Max. height: x when $v = 0$

$$= 180 \ln\left[\frac{253600}{3600 + 0^2}\right]$$

$$= 765.86..$$

$\doteqdot 766$m (nearest m).

(iii) $$\frac{dv}{dt} = \frac{-3600 - v^2}{360}$$

$$\therefore \frac{dt}{dv} = \frac{-360}{3600 + v^2}.$$

$$t = \int \frac{-360}{3600 + v^2}.dv$$

$$= -360 \times \frac{1}{60} \tan^{-1}\left(\frac{v}{60}\right) + C.$$

$$t = -6 \tan^{-1}\left(\frac{v}{60}\right) + C.$$

$t = 0.\ v = 500$

$$0 = -6 \tan^{-1}\left(\frac{500}{60}\right) + C$$

$$= -6 \tan^{-1}\left(\frac{25}{3}\right) + C.$$

$$6 \tan^{-1}\left(\frac{25}{3}\right) = C.$$

$$t = 6 \tan^{-1}\left(\frac{25}{3}\right) - 6 \tan^{-1}\left(\frac{v}{60}\right).$$

Time for max height is t when $v = 0$:

$$= 6 \tan^{-1}\left(\frac{25}{3}\right)$$

$$= 8.708..$$

Time to max. height $\doteqdot$ 8.7 seconds.

Q.(15)(a) (i) $x = 1 + 2\lambda$
$y = 10 - 3\lambda$
$z = 5 + \lambda$.

Note: Could also use
$\Delta = (-6)^2 - 4 \times 1 \times 9 = 0$
to show tangent.

$(x-5)^2 + (y+2)^2 + (z-3)^2 = 38.$
$(2\lambda - 4)^2 + (12 - 3\lambda)^2 + (2+\lambda)^2 = 38$
$14\lambda^2 - 84\lambda + 126 = 0$
$\lambda^2 - 6\lambda + 9 = 0$
$(\lambda - 3)^2 = 0$
$\Rightarrow$ Only 1 point of intersection: $\lambda = 3$.
$x = 1 + 2 \times 3 = 7.$
$y = 10 - 3 \times 3 = 1.$
$z = 5 + 3 = 8.$

Point of intersection $= \begin{pmatrix} 7 \\ 1 \\ 8 \end{pmatrix}$.

(ii) Point of intersection between

$$\begin{pmatrix} 1 \\ 10 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad \& \quad \begin{pmatrix} 15 \\ -1 \\ 6 \end{pmatrix} + \delta \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$$

$x: 1 + 2\lambda = 15 + 4\delta. \Rightarrow \lambda - 2\delta = 7 \ (1)$
$y: 10 - 3\lambda = -1 - \delta \quad \Rightarrow 3\lambda - \delta = 11 \ (2)$ +
$z: 5 + \lambda = 6 - \delta \Rightarrow \lambda + \delta = 1 \ (3)$
$4\lambda = 12$
$\lambda = 3.$

Sub. $\lambda = 3$ in (3): $3 + \delta = 1.$
$\delta = -2.$

Sub. $\lambda = 3, \delta = -2$ in (1) to make sure it works:
$3 - 2 \times -2 = 7$ ✓

Point of intersection $= \begin{pmatrix} 1 \\ 10 \\ 5 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 15 \\ -1 \\ 6 \end{pmatrix} - 2 \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 7 \\ 1 \\ 8 \end{pmatrix} \qquad = \begin{pmatrix} 7 \\ 1 \\ 8 \end{pmatrix}.$ ✓

PTO

(15)(a)(iii) Radius of sphere at $\begin{pmatrix}7\\1\\8\end{pmatrix}$

$$= \begin{pmatrix}7\\1\\8\end{pmatrix} - \begin{pmatrix}5\\-2\\3\end{pmatrix}$$

$$= \begin{pmatrix}2\\3\\5\end{pmatrix}$$

Showing $\begin{pmatrix}15\\-1\\6\end{pmatrix} + \delta\begin{pmatrix}4\\-1\\-1\end{pmatrix} \perp$ radius.

$$\begin{pmatrix}4\\-1\\-1\end{pmatrix} \cdot \begin{pmatrix}2\\3\\5\end{pmatrix}$$

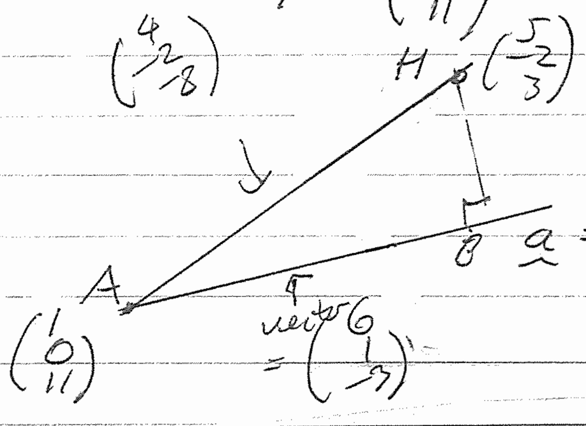
$$= 8 - 3 - 5$$

$$= 0.$$

As direction vector $\cdot$ radius $= 0$.

Line $\perp$ radius.

(iv) $\perp$ distance from $\begin{pmatrix}1\\0\\11\end{pmatrix} + \lambda\begin{pmatrix}6\\1\\-3\end{pmatrix}$ to $\begin{pmatrix}5\\-2\\3\end{pmatrix}$.



$$\text{Proj}_{\underset{\sim}{b}}\, \underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2}\, \underset{\sim}{b}$$

$$\underset{\sim}{a} = \begin{pmatrix}4\\-2\\-8\end{pmatrix} \quad \underset{\sim}{b} = \begin{pmatrix}6\\1\\-3\end{pmatrix}$$

$$= \frac{46}{46}\begin{pmatrix}6\\1\\-3\end{pmatrix} = \begin{pmatrix}6\\1\\-3\end{pmatrix}.$$

Then

$$\overrightarrow{HB} = \overrightarrow{HA} + \overrightarrow{AB}$$

$$= \overrightarrow{AB} - \overrightarrow{AH}$$

$$= \begin{pmatrix}6\\1\\-3\end{pmatrix} - \begin{pmatrix}4\\-2\\-8\end{pmatrix}$$

$$= \begin{pmatrix}2\\3\\5\end{pmatrix}$$

& $HB = |\overrightarrow{HB}| = \sqrt{2^2+3^2+5^2}$

$= \underline{\sqrt{38} \text{ units}}$

[Can also use Pythagoras].

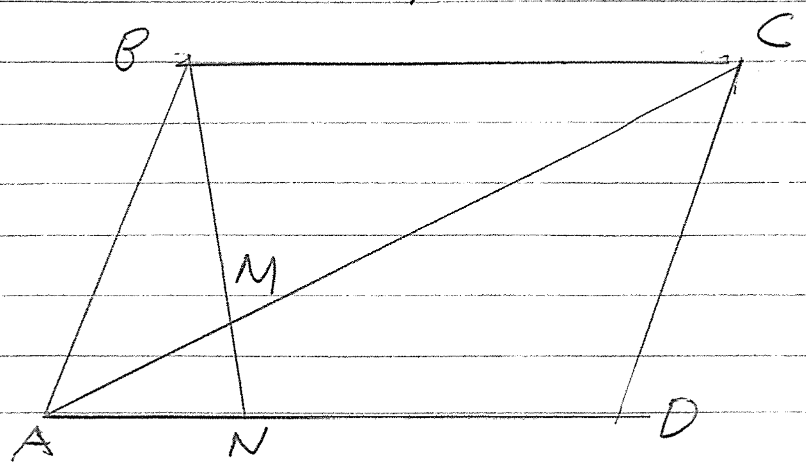
$AH = \sqrt{4^2+(-2)^2+8^2} = \sqrt{84}$.

$AB = \sqrt{6^2+1^2+(-3)^2} = \sqrt{46}$.

$HB = \sqrt{(\sqrt{84})^2 - (\sqrt{46})^2}$

$= \underline{\sqrt{38} \text{ units}}$.

(15)(b)



(i) $\underset{\sim}{AN} = \frac{1}{3}\underset{\sim}{q}$

$\therefore \underset{\sim}{NA} = -\frac{1}{3}\underset{\sim}{q}$.

$\underset{\sim}{NB} = \underset{\sim}{NA} + \underset{\sim}{AB}$

$= -\frac{1}{3}\underset{\sim}{q} + \underset{\sim}{p}$

$= \underset{\sim}{p} - \frac{1}{3}\underset{\sim}{q}$.

(ii) $\underset{\sim}{AM} = \lambda \underset{\sim}{AC}$

$\therefore \quad = \lambda(\underset{\sim}{p} + \underset{\sim}{q})$.

$\underset{\sim}{MB} = \delta \underset{\sim}{NB}$

$= \delta(\underset{\sim}{p} - \frac{1}{3}\underset{\sim}{q})$.

$\underset{\sim}{AB} = \underset{\sim}{AM} + \underset{\sim}{MB}$

$\underset{\sim}{p} + 0\underset{\sim}{q} = \lambda\underset{\sim}{p} + \lambda\underset{\sim}{q} + \delta\underset{\sim}{p} - \frac{1}{3}\delta\underset{\sim}{q}$.

$\underset{\sim}{p} + 0\underset{\sim}{q} = (\lambda+\delta)\underset{\sim}{p} + (\lambda - \frac{1}{3}\delta)\underset{\sim}{q}$.

Equating $\underset{\sim}{p}$: $\lambda + \delta = 1$. (1)

Equating q: $\lambda - \frac{1}{3}\delta = 0$ (2)

$\frac{4}{3}\delta = 1$.

$\delta = \frac{3}{4}$

As $\lambda + \delta = 1$, $\lambda = \frac{1}{4}$.

$\therefore \underset{\sim}{AM} = \frac{1}{4}\underset{\sim}{AC}$.

Q p.13

Q. (16)(a) (i) Euler's theorem:

$$z^n = e^{ni\theta} = \cos n\theta + i\sin n\theta.$$

$$z^{-n} = e^{-ni\theta} = \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos n\theta - i\sin n\theta \quad \text{(cos even, sin odd)}.$$

$$\therefore z^n + z^{-n} = 2\cos n\theta.$$

(ii) $$(z+z^{-1})^7 = z^7 + 7z^5 + 21z^3 + 35z + \frac{35}{z} + \frac{21}{z^3} + \frac{7}{z^5} + \frac{1}{z^7}$$

$$\left(z+\frac{1}{z}\right)^7 = \left(z^7+\frac{1}{z^7}\right) + 7\left(z^5+\frac{1}{z^5}\right) + 21\left(z^3+\frac{1}{z^3}\right) + 35\left(z+\frac{1}{z}\right)$$

$$\therefore (2\cos\theta)^7 = 2\cos 7\theta + 14\cos 5\theta + 42\cos 3\theta + 70\cos\theta.$$

$$128\cos^7\theta = 2\cos 7\theta + 14\cos 5\theta + 42\cos 3\theta + 70\cos\theta$$

$$\cos^7\theta = \frac{1}{64}\left[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\right]$$

as required.

(iii) $$\int_0^{\frac{\pi}{2}} \cos^7\theta \, d\theta = \frac{1}{64}\int_0^{\frac{\pi}{2}} \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta \, d\theta$$

$$= \frac{1}{64}\left[\frac{1}{7}\sin 7\theta + \frac{7}{5}\sin 5\theta + 7\sin 3\theta + 35\sin\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{64}\left(-\frac{1}{7} + \frac{7}{5} - 7 + 35\right)$$

$$= \frac{1}{64} \times \frac{1024}{35} = \boxed{\frac{16}{35}.}$$

(b)(i) $$I_n = \int_0^{\frac{\pi}{2}} \cos^n\theta \, d\theta \qquad u = \cos^{n-1}\theta \qquad v = \sin\theta$$

$$u' = (n-1)\cos^{n-2}\theta \cdot -\sin\theta \qquad v' = \cos\theta$$

By $\int uv' \, dx = uv - \int vu' \, dx$

$$\int_0^{\frac{\pi}{2}} \cos^n\theta \, d\theta = \left[\cos^{n-1}\theta\sin\theta\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2}\theta \cdot \sin^2\theta \, d\theta$$

$$I_n = 0 + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2}\theta(1-\cos^2\theta)\, d\theta$$

$$I_n = (n-1)I_{n-2} - (n-1)I_n.$$

$$nI_n = (n-1)I_{n-2}. \Rightarrow I_n = \frac{(n-1)}{n}I_{n-2}.$$

Q.(16)(b)(ii) $I_1 = \int_0^{\frac{\pi}{2}} \cos\theta \cdot d\theta$

$= [\sin\theta]_0^{\frac{\pi}{2}}$

$= 1.$

$I_7 = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 \; I_1$

$= \underline{\frac{16}{35}}.$

Q. (16)(c)(i) Hits target at (a, b).

$$y = -\frac{gx^2}{2V^2}(1+\tan^2\alpha) + x\tan\alpha$$

$$b = -\frac{ga^2}{2V^2} - \frac{ga^2}{2V^2}\tan^2\alpha + a\tan\alpha.$$

$\times 2V^2$ & $+2V^2b$.

$$0 = ga^2\tan^2\alpha - 2V^2a\tan\alpha + (ga^2+2V^2b).$$

$$\therefore \tan\alpha = \frac{2V^2a \pm \sqrt{4V^4a^2 - 4ga^2\times(ga^2+2V^2b)}}{2ga^2}$$

$$= \frac{2V^2a \pm \sqrt{4a^2[V^4 - g^2a^2 - 2gV^2b]}}{2ga^2}$$

$$= \frac{2V^2a \pm 2a\sqrt{V^4 - g^2a^2 - 2gV^2b}}{2ga^2} \quad \begin{matrix}\div 2a\\ \div 2a\end{matrix}$$

$$\tan\alpha. = \frac{V^2 \pm \sqrt{V^4 - g^2a^2 - 2gV^2b}}{ga.}$$

As $\alpha_2 > \alpha_1$, $\tan\alpha_2 = \dfrac{V^2 + \sqrt{V^4 - g^2a^2 - 2gV^2b}}{ga}$

& $\tan\alpha_1 = \dfrac{V^2 - \sqrt{V^4 - g^2a^2 - 2gV^2b}}{ga.}$

p16

(16)(c)(iii) $\tan(\alpha_1 + \alpha_2)$

$$= \frac{\tan\alpha_1 + \tan\alpha_2}{1 - \tan\alpha_1 \tan\alpha_2} \quad (1)$$

Note: $\tan\alpha_1 + \tan\alpha_2 = \dfrac{2V^2}{ga} \quad (2)$

& $\tan\alpha_1 \tan\alpha_2 = \dfrac{V^4 - (V^4 - g^2a^2 - 2gV^2b)}{g^2a^2}$

$$= \frac{g^2a^2 + 2gV^2b}{g^2a^2}$$

$$= 1 + \frac{2gV^2b}{g^2a^2}$$

$$= 1 + \frac{2V^2b}{ga^2}. \quad (3)$$

Sub. (2) & (3) in (1):

$$\tan(\alpha_1 + \alpha_2) = \frac{\frac{2V^2}{ga}}{1 - \left(1 + \frac{2V^2b}{ga^2}\right)}$$

$$= \frac{2V^2}{ga} \div \frac{-2V^2b}{ga^2}$$

$$= \frac{\cancel{2V^2}}{\cancel{ga}} \times \frac{-\cancel{ga^2}}{\cancel{2V^2}b}$$

$$= -\frac{a}{b}.$$

As $\tan\theta = \dfrac{b}{a}$,

$$\tan(90^\circ + \theta) = \tan[180^\circ - (90^\circ - \theta)]$$

$$= -\tan(90^\circ - \theta)$$

$$= -\frac{1}{\tan\theta}$$

$$= \frac{-a}{b}.$$

$\therefore \tan(\alpha_1 + \alpha_2) = \tan(90^\circ + \theta)$

$\therefore$ as $\alpha_1 + \alpha_2$ & $90^\circ + \theta$ both in Q2,

$\therefore \alpha_1 + \alpha_2 = 90^\circ + \theta$

$\quad -\theta - \alpha_2 \quad -\theta - \alpha_2$

$\alpha_1 - \theta = 90^\circ - \alpha_2$

$\angle BOA = \angle COY$

END OF SOLUTIONS!

(or)

(16) (c) (iii) alternative:

$\tan\alpha_1$ & $\tan\alpha_2$ are solutions to

$$ga^2\tan^2\alpha - 2V^2a\tan\alpha + (ga^2+2V^2b) = 0 \text{ [from p. 15]}.$$

By $\beta+\gamma = -\frac{b}{a}$ [sum of roots of quadratic equation]

$$\tan\alpha_1 + \tan\alpha_2 = \frac{2V^2a}{ga^2}$$

$$= \frac{2V^2}{ga}. \quad (1)$$

By $\beta\gamma = \frac{c}{a}$ [product of roots of quadratic equation]

$$\tan\alpha_1\tan\alpha_2 = \frac{ga^2+2V^2b}{ga^2}$$

$$= 1 + \frac{2V^2b}{ga^2}. \quad (2)$$

Hence by $\tan(\alpha_1+\alpha_2) = \frac{\tan\alpha_1 + \tan\alpha_2}{1-\tan\alpha_1\tan\alpha_2} \quad (3)$

sub. (1) & (2) in (3)

$$= \frac{\frac{2V^2}{ga}}{1-\left(1+\frac{2V^2b}{ga^2}\right)}$$

$$= \frac{2V^2}{ga} \div \frac{-2V^2b}{ga^2}$$

$$= \frac{2V^2}{ga} \times \frac{ga^2}{-2V^2b}$$

$$= -\frac{a}{b}$$

$$= \cot\theta$$

$$= \tan(90^\circ+\theta).$$